Estimating and Verifying the Lyapunov Spectrum of Models for Ship Rolling

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Abstract

As it is represented as a nonlinear equation of motion, the rolling of a ship is more difficult to predict than pitching and other rocking. On the other hand, because it is heavily involved in capsizing, its quantitative analysis is important when considering vessel stability. Therefore, this study aims to analyze the nonlinear nature of the equation of motion that represents the rolling of a ship, in order to gain knowledge on vessel capsize phenomena. In this study, use of the Lyapunov spectrum enabled the extraction of the unstable behavior of the system and facilitated the ascertaining of motion characteristics. Proof was thus obtained that the Lyapunov spectrum's special property of quantifying trajectory instability is useful in analyzing motion models for rolling.

1 Introduction

It is well known that in the rolling of ships, a strong nonlinearity arises between the external force (the input, represented by waves) and rolling (the output). Due to this nonlinearity, rolling is more difficult to estimate than other kinds of motion, such as pitching. That said, rolling plays a major role in capsizing, and so it is important to analyze it quantitatively in considering the safety of ships.To gain insight into the capsize phenomenon, this study aims to analyze the nonlinearity in equations of motion that express ship rolling.

Some attempts have already been made to analyze modes of motion numerically or geometrically [1, 2, 3], which include studies that analyze ship motion and the capsize phenomenon using Lyapunov exponents and local Lyapunov exponents, which are known for their use in chaos analysis. This indicates the usefulness of the Lyapunov exponent in analyzing the capsize phenomenon[4]. In this paper, we will use the Lyapunov spectrum in a Duffing capsize model, which was also used in research by Kan, Taguchi, et al., to examine whether the Lyapunov spectrum is also usable and useful in simpler models.

2 Capsize model

The equation of motion expressing ship rolling is generalized in the following form, which incorporates a nonlinear term with a negative coefficient in the righting term:

$$I \cdot \frac{d^2\phi}{dt^2} + N \cdot \frac{d\phi}{dt} + W \cdot GM \cdot \phi\{1 - (\phi/\phi_\nu)^2\} = M_0 + M_r \cos(\omega t + \delta) \tag{2.1}$$

where ϕ is the rolling angle, ϕ_ν is the angle of vanishing stability, I is the rolling moment of inertia, N is the damping force coefficient, M_r is the exciting force coefficient, W is the weight of displacement, ω is the frequency of wave encounter, t is time, and δ is the exciting force phase.

Consequently, it is possible to represent ship rolling and the capsize phenomenon by making a mathematical model of the equation of motion, which consists of an inertia term, a damping term, a righting term, and an exciting term.

Furthermore, given the natural rolling frequency when upright $\omega_0 = (W \cdot GM/I)^{\frac{1}{2}}$, time $s = \omega_0 t$, and rolling angle $\psi = \phi/\phi_\nu$ if we make the corresponding substitutions in Expression (2.1), we can make the equation dimensionless and simplify it as follows:

$$\frac{d^2\psi}{ds^2} + \nu\frac{d\psi}{ds} + \psi - \psi^3 = B_0 + B\cos(\Omega s + \varepsilon) \tag{2.2}$$

where $\nu = N/I\omega_0$, $B_0 = M_0/I{\omega_0}^2\phi_\nu$, $B = M_r/I{\omega_0}^2\phi_\nu$ and $\Omega = \omega/\omega_0$. In this paper, we will pay particular attention to cases where $B_0 = 0$ or $\varepsilon = 0$.

Expression (2.2) is a so-called soft-spring Duffing system. It is well-known that Duffing equations can lead to chaos depending on their parameters and initial values, and the Japanese attractor discovered by Ueda (Ueda's attractor) is famous [5]. A soft-spring system can exhibit not only periodic motion and chaotic motion, but also an "explosion of solutions," which means the dispersion of values within a limited time. This corresponds to capsizing in a model of ship rolling.

3 Lyapunov spectrum

The Lyapunov exponent is an index that quantifies orbital instability. It quantifies the rate of expansion according to the time evolution of the microscopic displacement of two points that are extremely close to each other in a dynamical system. The microscopic displacement δx_t at time t can be expressed as

$$\delta x_t = \delta x_0 \exp(\lambda t) \tag{3.1}$$

where λ is the Lyapunov exponent. Note that δx_0 is the difference in orbit at $t = 0$. Rewriting the above equation (3.1) in terms of λ gives us

$$\lambda = \frac{1}{t}\log\left|\frac{\delta x_t}{\delta x_0}\right| \tag{3.2}$$

Furthermore, there is one Lyapunov exponent for each dimension of the system, and the set of these is called the Lyapnov spectrum. When one or more Lyapunov exponents have a positive value, this indicates that the system has orbital instability. Thus, as long as there is no noise or other probabilistic factors in the original data, the system can be regarded as chaotic. Conversely, when the system is in quasi-periodic motion, all Lyapunov exponents have a zero or negative value. Since whether the values in the Lyapunov spectrum are positive or negative generally corresponds to the behavior of the system, it is possible to estimate the characteristics of the system by finding the Lyapunov spectrum. For example, a three-dimensional dynamical system is in quasi-periodic motion when the combination of symbols for its Lyapunov spectrum is $(0, -, -)$ and chaotic motion when the combination is $(+, 0, -)$.

It is difficult to solve for Lyapunov exponents using the definition in Expression (3.2). Calculators are limited in their precision, so when calculating an actual Lyapunov spectrum, we need a way to determine the change according to the time evolution of the microscopic displacement δx_t.

4 Methods of estimating the Lyapunov spectrum in known models

The calculations in this study were based on the algorithms presented in Wolf's paper [6]. In this method, the change in the initial condition caused by the perturbation added to the underlying model $\dot{\boldsymbol{x}}_i = \boldsymbol{F}_i(\boldsymbol{x}_i)(1 \leq i \leq N)$ is represented by linear approximation using a Jacobian matrix $\boldsymbol{J}(t)$.

$$\delta\dot{\boldsymbol{x}}_i(t) = \boldsymbol{J}(t)\delta\boldsymbol{x}_i(t) \tag{4.1}$$

Then, by solving the differential equation that expresses this microscopic change, we get the displacement vector after time τ, $\delta\boldsymbol{x}_i(t+\tau)$. As a rule, by solving Expression (4.1), we can evolve the system and measure the elongation $\delta\boldsymbol{x}_i$ for each unit of time to calculate the Lyapunov spectrum. However, if we keep on calculating $\delta\boldsymbol{x}_i(t+\tau)$ like this, its value will be squashed in the direction of contraction and ultimately become impossible to calculate, so we will use the following Gram-Schmidt orthonormalization to prevent this:

$$\boldsymbol{e}_i(t+\tau) = \delta\boldsymbol{x}_i(t+\tau) - \sum_{j=1}^{i-1} \langle \delta\boldsymbol{x}_i(t+\tau), \delta\boldsymbol{x}_j^{'}(t+\tau)\rangle \delta\boldsymbol{x}_j^{'}(t+\tau) \tag{4.2}$$

$$\delta\boldsymbol{x}_i^{'}(t+\tau) = \frac{\boldsymbol{e}_i(t+\tau)}{|\boldsymbol{e}_i(t+\tau)|} \tag{4.3}$$

where $\langle\ ,\ \rangle$ represent the inner product.

Based on Expression (4.2), we will orthogonalize $\delta\boldsymbol{x}_i(t+\tau)$ at each unit of time τ to find $\boldsymbol{e}_i(t+\tau)$. We will then set $\delta\boldsymbol{x}_i^{'}(t+\tau)$, in which $\boldsymbol{e}_i(t+\tau)$ is normalized, as our new vector of microscopic displacement. We can repeat this process, substituting the $\boldsymbol{e}_i(t)$ series obtained from evolving Expression (4.1) into Expression (4.4), to find the Lyapunov spectrum.

$$\lambda_i = \lim_{t\to\infty} \frac{1}{N\tau} \sum_{t=0}^{N-1} \log|\boldsymbol{e}_i(t)| \tag{4.4}$$

Here, $\lambda_i \geq \lambda_{i+1}$. For the series of numerical calculations, we used the four-dimensional Runge-Kutta method to estimate the Lyapunov spectrum with step size $\tau = 0.01$.

To find the Lyapunov spectrum from a ship capsize model, it is neccessary to convert it to a self-excited differential equation and write it using three expressions.

$$\begin{aligned} \frac{d\psi}{ds} &= u \\ \frac{du}{ds} &= -\nu u - \psi + \psi^3 + B\cos\theta \\ \frac{d\theta}{ds} &= \Omega \end{aligned} \tag{4.5}$$

5 Results of Lyapunov spectrum analysis

The time-series waveform of the forced Duffing equation and an example of the estimated Lyapunov spectrum are shown below.

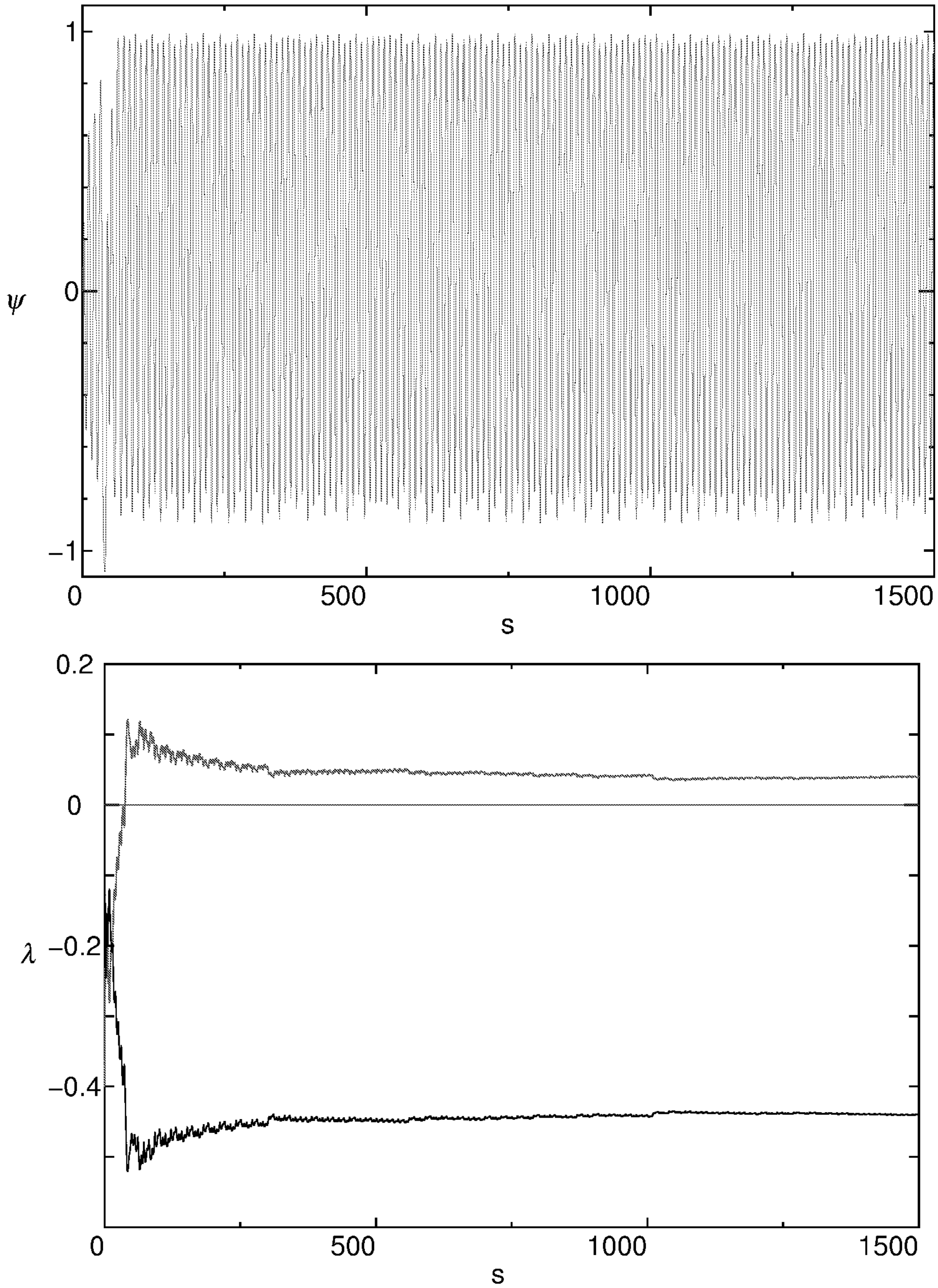

Figure 1: Chaotic motion($\nu = 0.4, B = 0.263, \Omega = 0.63$)

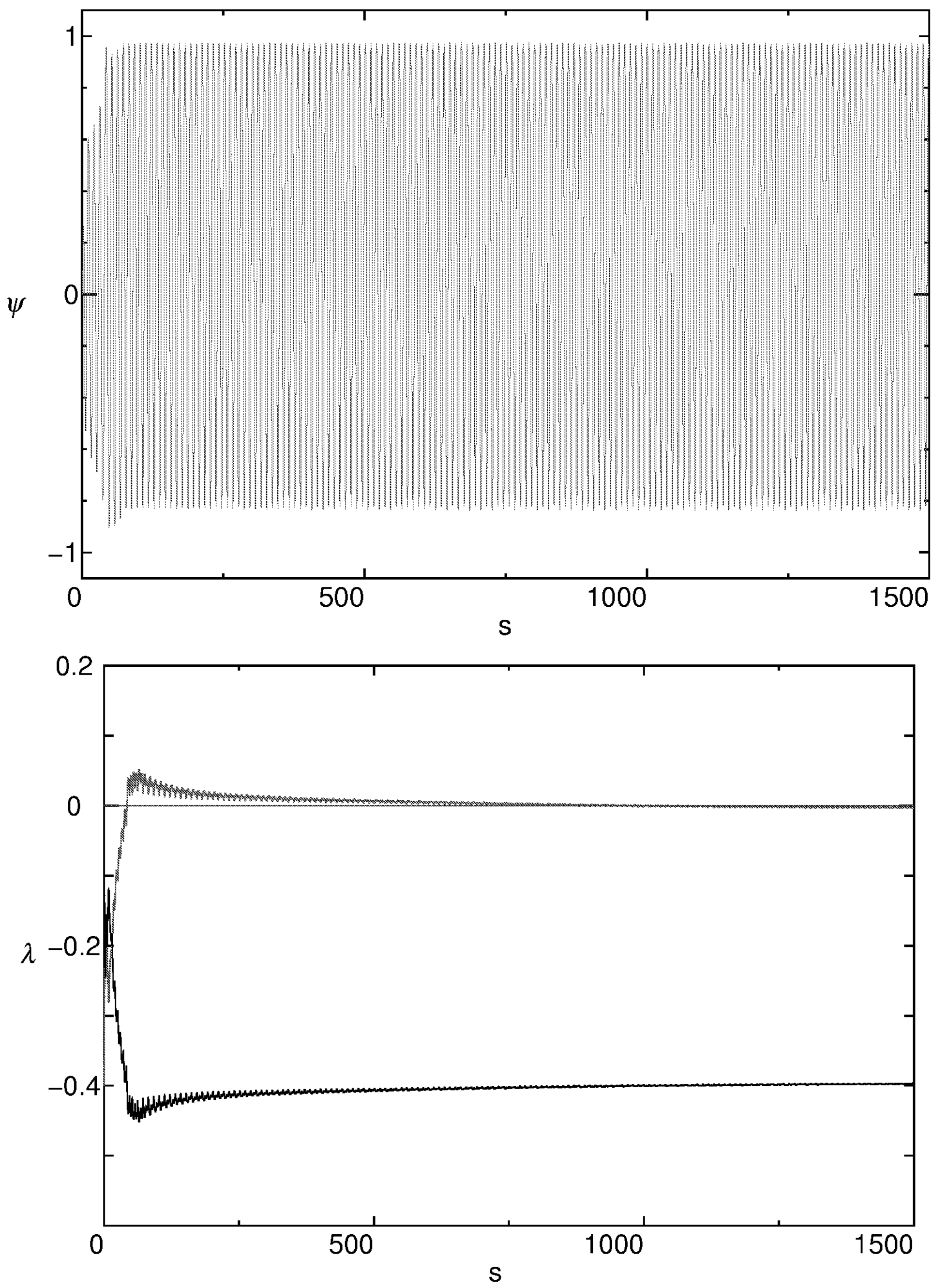

Figure 2: Quasi-periodic motion($\nu = 0.4, B = 0.261, \Omega = 0.63$)

Figure. 1 illustrates one case where the orbit is not periodic. The set of values in the Lyapunov spectrum is $(+, 0, -)$, which indicates quantitatively that the behavior of the system is chaotic. Here, the Lyapunov dimension, a kind of fractal dimension, is 2.092. Figure. 2, meanwhile, illustrates one case where the orbit is quasi-periodic. All of the values in the Lyapunov spectrum are zero or negative, reflecting the characteristics of the system. One of the Lyapunov exponents did assume a positive value in the interval up to 500 s, but this was found to reflect the transient state of the underlying rolling waveform.

6 Conclusion

Through the above prosess, we were able to estimate the Lyapunov spectrum of a Duffing capsize model using the algorithm presented in Wolf's paper.

Chaotic behavior in a soft-spring Duffing system is difficult to discern through a time-series waveform alone; it is thought that one cannot see it without representing it on a phase plane or Poincaré section [2]. However, using the Lyapunov spectrum enabled us to extract the unstable behavior exhibited by the system and made it easy to understand the characteristics of its motion. Compared to using a phase plane or Poincaré section, the Lyapunov spectrum also enables comparisons with multiple results because it quantifies the shape of the valus.

For these reasons, the Lyapunov spectrum's characteristic of quantifying orbital instability was proven to be useful in analyzing models for the rolling of ships. In future research, we hope to use the Lyapunov spectrum to analyze the capsize phenomenon.

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